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Uniqueness for a Class of Spatially Homogeneous Boltzmann Equations Without Angular Cutoff

Nicolas Fournier[1]

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We consider the 3-dimensional spatially homogeneous Boltzmann equation, which describes the evolution in time of the velocity distribution in a gas, where particles are assumed to undergo binary elastic collisions. We consider a cross section bounded in the relative velocity variable, without angular cutoff, but with a moderate angular singularity. We show that there exists at most one weak solution with finite mass and momentum. We use a Wasserstein distance. Although our result is far from applying to physical cross sections, it seems to be the first one which deals with cross sections without cutoff for non Maxwellian molecules.

KEY WORDS: Boltzmann equation, Uniqueness, Wasserstein distance
MSC 2000 : 82C40.

1. INTRODUCTION

We consider a spatially homogeneous 3-dimensional gas. Let $f(t, v)$ stand for the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. Then f solves the Boltzmann equation

$$\partial_t f(t, v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma B(|v - v_*|, \theta)[f(t, v')f(t, v'_*) - f(t, v)f(t, v_*)], \tag{1.1}$$

where $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$ and θ is the angle between $v - v_*$ and σ. The *cross section* $B(|v - v_*|, \theta) = B(|v' - v'_*|, \theta)$ represents the rate at which two particles with velocities v' and v'_* (resp. v and v_*) collide in such a way that the resulting particles have the velocities v and v_* (resp. v' and v'_*). We refer to Desvillettes[(3)] and Villani[(14)] for many details on what is known on this topic.

[1] Centre de Mathématiques, Faculté de Sciences et Technologie, Université Paris XII, 61 avenue du Général de Gaulle 94 010, Créteil Cedex, France; e-mail: nicolas.fournier@univ-paris12.fr

Conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1), that is for all $t \geq 0$,

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,v)dv = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(0,v)\,dv, \tag{1.2}$$

and we classically may assume without loss of generality that $\int_{\mathbb{R}^3} f(0,v)dv = 1$. Another important *a priori* estimate is the decrease of entropy: for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f(t,v)\log f(t,v)\,dv \leq \int_{\mathbb{R}^3} f(0,v)\log f(0,v)\,dv. \tag{1.3}$$

For example, the solutions built in Goudon[7] or Villani[13] enjoy these properties.

We consider here the problem of uniqueness. We will assume that the cross section is of the form $B(|v-v_*|,\theta)\sin\theta = b(|v-v_*|)\beta(\theta)$. From the physical point of view, one usually assumes that $b(|v-v_*|) \simeq |v-v_*|^\gamma$ for some $\gamma \in (-\infty, 1]$, that $\int_0^\pi \beta(\theta)d\theta = \infty$, but $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$. Hence β should be allowed to explode near 0, which expresses the predominance of *grazing collisions*.

In the case with angular cutoff, that is when $\int_0^\pi \beta(\theta)d\theta < \infty$, there are some good uniqueness results, see Mischler-Wennberg.[9]

All the available results in the case without cutoff concern Maxwellian molecules ($b \equiv 1$). The first result was that of Tanaka,[10] who proved the uniqueness of the solution to a martingale problem associated with the Boltzmann equation (assuming that $b \equiv 1$, $\int_0^\pi \theta\beta(\theta)d\theta < \infty$ and $\int_{\mathbb{R}^3}(1+|v|)f(0,dv) < \infty$). Horowitz-Karandikar[8] obtained the first true uniqueness result for the Boltzmann equation, assuming that $b \equiv 1$, $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$ and $\int_{\mathbb{R}^3}(1+|v|^2)f(0,dv) < \infty$. Under the same assumptions, Toscani-Villani[12] were able to give a much simpler proof. All the previously cited works deal with measure solutions.

We aim here to prove the uniqueness (and continuous dependance in the initial condition) of solutions for some cross sections of the form $B(|v-v_*|,\theta)\sin\theta = \min(|v-v_*|^\gamma, A)\,\beta(\theta)$, with $\gamma \in (-\infty,\infty)$, $A \in (0,\infty)$ and with $\int_0^\pi \theta\beta(\theta)d\theta < \infty$. In practise, this assumption imposes an upperbound on the relative velocity. Similar unphysical assumptions have been often used in kinetic theory to avoid problems in proofs, see for example Arkeryd and Nouri.[1] Our result holds for function solutions having finite mass and momentum. We use a Wasserstein metric. Our method is inspired by the works of Tanaka,[10] Horowitz-Karandikar[8] and Desvillettes-Graham-Méléard.[4]

Let us finally mention that in a work in preparation, Desvillettes-Mouhot[5] obtain some new uniqueness results. It seems that their result applies to the (much more realistic) case $b(|v-v_*|) = |v-v_*|^\gamma$, $\gamma \in [0,1]$, with also a moderate angular singularity $\int_0^\pi \theta\beta(\theta)d\theta < \infty$. They work in Sobolev spaces with weights, so that they assume much more regularity on the initial condition.

We present our main result in Sec. 2 and handle the proof in the next sections.

2. MAIN RESULT

In this section, we first write in a suitable weak form the Boltzmann equation, we introduce the distance we will use and we state our main result.

We denote by $Lip(\mathbb{R}^3)$ the set of globally Lipschitz functions $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$. We consider the set $\mathcal{P}(\mathbb{R}^3)$ of probability measures on $\mathbb{R}^3$, and

$$\mathcal{P}_1(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3),\ m_1(f) < \infty\}, \quad m_1(f) = \int_{\mathbb{R}^3} |v| f(dv). \tag{2.1}$$

A measurable family $(f(t))_{t\geq 0}$ of probability measures on $\mathbb{R}^3$ is said to belong to $L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ if for all $T \in [0,\infty)$, $\sup_{[0,T]} m_1(f(t)) < \infty$. We denote, by $\mathcal{P}_1^d(\mathbb{R}^3)$ the set of probability measures $f \in \mathcal{P}_1(\mathbb{R}^3)$ having a density with respect to the Lebesgue measure. A family $(f(t))_{t\geq 0} \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ is said to belong to $L^\infty_{loc}(\mathcal{P}_1^d(\mathbb{R}^3))$ if for each $t \geq 0$, $f(t)$ has a density.

For $v, v_* \in \mathbb{R}^3$, and for $\sigma \in S^2$, we write

$$v' = v'(v, v_*, \sigma) = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \tag{2.2}$$

and we denote by θ, φ the colatitude and longitude of σ in some spherical coordinates with polar axis $(v - v_*)$.

We will consider the following class of cross sections.

Assumption (H)**.** The cross section is of the form $B(z,\theta)\sin\theta = b(z)\beta(\theta)$ for some functions $b : [0,\infty) \mapsto [0,\infty)$ and $\beta : [-\pi,\pi]\backslash\{0\} \mapsto [0,\infty)$. Furthermore, $\kappa_1 = \int_0^\pi \theta\beta(\theta)d\theta < \infty$. Next, b is continuous and bounded by a constant κ_2. Finally, there exists a constant κ_3 such that for all $x, y \in [0,\infty)$,

$$x(b(x) - b(y))_+ \leq \kappa_3|x - y|, \quad \text{where} \quad (b(x) - b(y))_+ = \max[b(x) - b(y), 0]. \tag{2.3}$$

Remark 2.1. *We are still far from general physical assumptions. Note however that that for any $\gamma \in (-\infty, +\infty)$, any $A \in (0,\infty)$, $B(z,\theta)\sin\theta = \min(z^\gamma, A)\beta(\theta)$ fullfills (H) as soon as $\int_0^\pi \theta\beta(\theta)d\theta < \infty$.*

We now define the notion of weak solutions we will use.

Definition 2.2. *A family of probability measures $f = (f(t))_{t\geq 0} \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ is said to be a weak solution to (1.1) if for any $\phi \in Lip(\mathbb{R}^3)$,* *any* $t \geq 0$,

$$\int_{\mathbb{R}^3} \phi(v) f(t, dv) = \int_{\mathbb{R}^3} \phi(v) f(0, dv) + \int_0^t ds \int_{\mathbb{R}^3} f(s, dv) \int_{\mathbb{R}^3} f(s, dv_*) A\phi(v, v_*), \quad (2.4)$$

where

$$A\phi(v, v_*) = b(|v - v_*|) \int_0^{\pi} d\theta \beta(\theta) \int_0^{2\pi} d\varphi [\phi(v') - \phi(v)]. \quad (2.5)$$

Note that for any $\sigma \in S^2$,

$$|v' - v| \leq |v - v_*| \sqrt{\frac{1 - \cos\theta}{2}} \leq \frac{\theta}{2} |v - v_*|, \quad (2.6)$$

so that that under (H) and for $f \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$, all the terms in (2.4) are well-defined. Let us introduce the distance on $\mathcal{P}_1(\mathbb{R}^3)$ we will use.

Notation 2.3. *For* $g, \tilde{g} \in \mathcal{P}_1(\mathbb{R}^3)$*, let* $\mathcal{H}(g, \tilde{g})$ *be the set of probability measures* G *on* $\mathbb{R}^3 \times \mathbb{R}^3$ *with marginals* $\int_{\tilde{v} \in \mathbb{R}^3} G(dv, d\tilde{v}) = g(dv)$ *and* $\int_{v \in \mathbb{R}^3} G(dv, d\tilde{v}) = \tilde{g}(d\tilde{v})$*. We then set*

$$d_1(g, \tilde{g}) = \inf \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}| G(dv, d\tilde{v}), \quad G \in \mathcal{H}(g, \tilde{g}) \right\}. \quad (2.7)$$

One may check that the inf is actually a min, and that for $g, g_n \in \mathcal{P}_1(\mathbb{R}^3)$, $\lim_n d_1(g_n, g) = 0$ if and only if for any $\phi \in Lip(\mathbb{R}^3)$, $\lim_n \int \phi(v) g_n(dv) = \int \phi(v) g(dv)$. We refer to Villani [15, Sections 7.1 and 7.2] for more details on this distance.

Theorem 2.4. *Assume* (H)*. For any pair* $f, \tilde{f} \in L^\infty_{loc}(\mathcal{P}_1^d(\mathbb{R}^3))$ *of weak solutions to (1.1), any* $t \geq 0$,

$$d_1(f(t), \tilde{f}(t)) \leq d_1(f(0), \tilde{f}(0)) e^{6\pi\kappa_1(2\kappa_2 + \kappa_3)t}. \quad (2.8)$$

Thus for any $f_0 \in \mathcal{P}_1^d(\mathbb{R}^3)$*, there exists at most one weak solution* $f \in L^\infty_{loc}(\mathcal{P}_1^d(\mathbb{R}^3))$ *to (1.1).*

Notice here that since d_1 is well-defined on $\mathcal{P}_1(\mathbb{R}^3)$, and since the constants in the right hand side of (2.8) do not depend on $f, \tilde{f}$, our result should extend to any pair $f, \tilde{f} \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ of weak solutions. However, a lack of continuity in the parameterization we use (see Notation 3.1 below) enforces us to assume that f

and $\tilde{f}$ admit densities. This restriction could be removed for the 2-dimensional Boltzmann equation.

The rest of the paper is dedicated to the proof of this result: we state some preliminaries and prove our main result in Sec. 3, and conclude in Sec. 4 with the proofs of preliminaries.

3. PROOF

Consider two weak solutions f and $\tilde{f}$ to (1.1). The main idea of the proof is to build a family $(F(t))_{t\geq 0}$ of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ such that for each $t \geq 0$, the marginals of $F(t)$ are $f(t)$ and $\tilde{f}(t)$, and in such a way that $F(t, dv, d\tilde{v})$ is supported as much as possible around the diagonal $v = \tilde{v}$. To do so, we will build a sort a *coupled* Boltzmann equation, of which the solution F has as marginals f and $\tilde{f}$. The dynamics of F has to *modify* its two marginal in a very similar (or coupled) way. We thus have to parameterize precisely the post-collisional velocities. We follow here the approach of Ref. 6, which was strongly inspired by Tanaka.[10]

Notation 3.1. *(i) For $X \in \mathbb{R}^3\backslash\{0\}$, we set $I(X) = \frac{|X|}{\sqrt{X_x^2+X_y^2}}(-X_y, X_x, 0)$ if $X_x^2 + X_y^2 > 0$ and $I(X) = (X_z, 0, 0)$ else. We also build $J(X) = \frac{X}{|X|} \wedge I(X)$. Then for each $X \in \mathbb{R}^3$, $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is a direct orthonormal basis of $\mathbb{R}^3$. Furthermore, I and J are almost everywhere continuous. (ii) For $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$, and $X, v, v_* \in \mathbb{R}^3$, we introduce*

$$\Gamma(X, \varphi) = (\cos\varphi)I(X) + (\sin\varphi)J(X),$$

$$\begin{aligned} v' = v'(v, v_*, \theta, \varphi) &= v + a(v, v_*, \theta, \varphi) \\ &= v + \frac{\cos\theta - 1}{2}(v - v_*) + \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi). \end{aligned} \quad (3.1)$$

One may then write, for all $\phi \in Lip(\mathbb{R}^3)$, any $\varphi_0 \in [0, 2\pi]$ (which may depend on v, v_*, θ),

$$\begin{aligned} A\phi(v, v_*) &= b(|v - v_*|)\int_0^\pi d\theta\beta(\theta)\int_0^{2\pi} d\varphi[\phi(v'(v, v_*, \theta, \varphi)) - \phi(v)] \\ &= b(|v - v_*|)\int_0^\pi d\theta\beta(\theta)\int_0^{2\pi} d\varphi[\phi(v'(v, v_*, \theta, \varphi + \varphi_0)) - \phi(v)], \end{aligned} \quad (3.2)$$

We use here and below the abusive notation $\varphi + \varphi_0 = \varphi + \varphi_0$ [mod 2π].

One problem with this parameterization is the lack of smoothness of the maps $X \mapsto I(X)$ and $X \mapsto J(X)$. To overcome this difficulty, we will use the following fine version of a Lemma due to Tanaka,[10] of which the proof may be found in Ref. 6, Lemma 2.6 (the almost everywhere continuity is not stated in Ref. 6 but is clear from the proof).

Lemma 3.2. *There exists an almost everywhere continuous function $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$ such that for all $X, Y \in \mathbb{R}^3$, all $\varphi \in [0, 2\pi]$,*

$$|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \leq 3|X - Y|. \tag{3.3}$$

This implies that for all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$, all $\theta \in [0, \pi]$, all $\varphi \in [0, 2\pi]$,

$$\begin{aligned} &|a(v, v_*, \theta, \varphi) - a(\tilde{v}, \tilde{v}_*, \theta, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)| \\ &\leq 2\theta \left(|v - \tilde{v}| + |v_* - \tilde{v}_*|\right). \end{aligned} \tag{3.4}$$

We now define a *coupled* version of the operator A.

Lemma 3.3. *Recall that $x \wedge y = \min(x, y)$ and $x_+ = \max(x, 0)$. For $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$, and $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, we set $v' = v'(v, v_*, \theta, \varphi)$ and $\tilde{v}' = v'(\tilde{v}, \tilde{v}_*, \theta, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))$. For $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$, we set*

$$\begin{aligned} &\mathcal{A}\psi(v, v_*, \tilde{v}, \tilde{v}_*) \\ &= \int_0^\pi d\theta\beta(\theta) \int_0^{2\pi} d\varphi \left\{ \begin{array}{l} [b(|v - v_*|) \wedge b(|\tilde{v} - \tilde{v}_*|)]\,[\psi(v', \tilde{v}') - \psi(v, \tilde{v})] \\ \quad + [b(|v - v_*|) - b(|\tilde{v} - \tilde{v}_*|)]_+\,[\psi(v', \tilde{v}) - \psi(v, \tilde{v})] \\ \quad + [b(|\tilde{v} - \tilde{v}_*|) - b(|v - v_*|)]_+\,[\psi(v, \tilde{v}') - \psi(v, \tilde{v})] \end{array} \right\}. \end{aligned} \tag{3.5}$$

Assume (H). Then $\mathcal{A}\psi$ is well-defined for all $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$. Furthermore,

(i) for $\phi \in Lip(\mathbb{R}^3)$, if $\psi(v, \tilde{v}) = \phi(v)$ (resp. $\psi(v, \tilde{v}) = \phi(\tilde{v})$), then $\mathcal{A}\psi(v, v_*, \tilde{v}, \tilde{v}_*) = A\phi(v, v_*)$ (resp. $\mathcal{A}\psi(v, v_*, \tilde{v}, \tilde{v}_*) = A\phi(\tilde{v}, \tilde{v}_*)$),

(ii) if $\psi(v, \tilde{v}) = |v - \tilde{v}|$, then

$$|\mathcal{A}\psi(v, v_*, \tilde{v}, \tilde{v}_*)| \leq 2\pi\kappa_1(2\kappa_2 + \kappa_3)\left(|v - \tilde{v}| + |v_* - \tilde{v}_*|\right). \tag{3.6}$$

Point (ii) expresses a sort of continuity in the pre-collisional velocities v, v_* of the averaged post-collisonal velocities (for the Boltzmann equation dynamics). It is, in some sense, the central argument of our proof.

Proof: First, $\mathcal{A}\psi$ is well-defined for $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$ due to (H) and (2.6). Point (i) is straightforward from the fact that for $x, y \in \mathbb{R}$, $x \wedge y + (x - y)_+ = x$, and from (3.2). We now prove point (ii). Using (H), (3.1) and Lemma 3.2, we obtain

$$\begin{aligned} &[b(|v-v_*|)\wedge b(|\tilde{v}-\tilde{v}_*|)]\,||v'-\tilde{v}'|-|v-\tilde{v}|| \\ &\quad\le \kappa_2\,|a(v,v_*,\theta,\varphi)-a(\tilde{v},\tilde{v}_*,\theta,\varphi+\varphi_0(v-v_*,\tilde{v}-\tilde{v}_*))| \\ &\quad\le 2\kappa_2\theta(|v-\tilde{v}|+|v_*-\tilde{v}_*|). \end{aligned} \tag{3.7}$$

Using now (H), (3.1) and (2.6),

$$\begin{aligned} &[b(|v-v_*|)-b(|\tilde{v}-\tilde{v}_*|)]_+\,||v'-\tilde{v}|-|v-\tilde{v}|| \\ &\quad\le [b(|v-v_*|)-b(|\tilde{v}-\tilde{v}_*|)]_+\,|v'-v| \\ &\quad\le \frac{\theta}{2}\,[b(|v-v_*|)-b(|\tilde{v}-\tilde{v}_*|)]_+\,|v-v_*| \\ &\quad\le \frac{\theta}{2}\kappa_3||v-v_*|-|\tilde{v}-\tilde{v}_*|| \\ &\quad\le \frac{\theta}{2}\kappa_3(|v-\tilde{v}|+|v_*-\tilde{v}_*|). \end{aligned} \tag{3.8}$$

By the same way,

$$[b(|\tilde{v}-\tilde{v}_*|)-b(|v-v_*|)]_+\,||v-\tilde{v}'|-|v-\tilde{v}|| \le \frac{\theta}{2}\kappa_3(|v-\tilde{v}|+|v_*-\tilde{v}_*|). \tag{3.9}$$

The conclusion follows, recalling that $\kappa_1=\int_0^\pi \theta\beta(\theta)d\theta<\infty$. □

We will need some auxilliary results: first, the uniqueness for a linearized Boltzmann equation, that will be proved in the next section.

Proposition 3.4. *Assume (H). Let $f=(f(t))_{t\ge0}\in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ be a weak solution to (1.1). Consider a family $g=(g(t))_{t\ge0}\in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ such that for any $\phi\in Lip(\mathbb{R}^3)$, any $t\ge0$,*

$$\begin{aligned} \int_{\mathbb{R}^3}\phi(v)g(t,dv) &= \int_{\mathbb{R}^3}\phi(v)f(0,dv) \\ &\quad+\int_0^t ds\int_{\mathbb{R}^3} g(s,dv)\int_{\mathbb{R}^3} f(s,dv_*)A\phi(v,v_*). \end{aligned} \tag{3.10}$$

Then $g \equiv f$.

We define $\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$, $\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3)$, $L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$, $\mathcal{P}^d_1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $L^\infty_{loc}(\mathcal{P}^d_1(\mathbb{R}^3 \times \mathbb{R}^3))$ as in the case of $\mathbb{R}^3$. The following existence result for a *coupled* linearized Boltzmann equation will be checked in the next section.

Proposition 3.5. *Assume* (H). *Consider a given family of probability measures* $F = (F(t))_{t\geq 0} \in L^\infty_{loc}(\mathcal{P}^d_1(\mathbb{R}^3 \times \mathbb{R}^3))$. *We say that a family of probability measures* $G = (G(t))_{t\geq 0} \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$ *is a weak solution to* $(LCB(F))$ *if for any* $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$, *any* $t \geq 0$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \psi(v,\tilde{v})G(t,dv,d\tilde{v}) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \psi(v,\tilde{v})F(0,dv,d\tilde{v}) + \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G(s,dv,d\tilde{v}) \int_{\mathbb{R}^3\times\mathbb{R}^3} F(s,dv_*,d\tilde{v}_*)\mathcal{A}\psi(v,v_*,\tilde{v},\tilde{v}_*). \quad (3.11)$$

There exists at least one weak solution G *to* $(LCB(F))$.

Finally, we will need the following Lemma, that will also be checked in the next section.

Lemma 3.6. *Recall Notation 2.3, and consider two probability measures with densities* $g, \tilde{g} \in \mathcal{P}^d_1(\mathbb{R}^3)$. *Then, if* $\mathcal{H}_d(g,\tilde{g}) = \mathcal{H}(g,\tilde{g}) \cap \mathcal{P}^d_1(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$d_1(g,\tilde{g}) = \inf\left\{\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|G(dv,d\tilde{v}),\ G \in \mathcal{H}_d(g,\tilde{g})\right\}. \quad (3.12)$$

We may now prove our main result.

Proof of Theorem 2.4. We consider two weak solutions $f, \tilde{f} \in L^\infty_{loc}(\mathcal{P}^d_1(\mathbb{R}^3))$ to (1.1), and divide the proof into three steps.

Step 1. Recall Notation 2.3, and fix $\varepsilon > 0$. For each $t \geq 0$, $f(t)$ and $\tilde{f}(t)$ are assumed to admit some densities, so that due to Lemma 3.6, we may find $F^\varepsilon(t) \in \mathcal{H}_d(f(t),\tilde{f}(t))$ such that

$$d_1(f(t),\tilde{f}(t)) \leq \int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|F^\varepsilon(t,dv,d\tilde{v}) \leq d_1(f(t),\tilde{f}(t)) + \varepsilon. \quad (3.13)$$

Then clearly, $F^\varepsilon = (F^\varepsilon(t))_{t\geq 0}$ belongs to $L^\infty_{loc}(\mathcal{P}^d_1(\mathbb{R}^3 \times \mathbb{R}^3))$. Recalling Proposition 3.5, we consider a solution $G^\varepsilon \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$ to $(LCB(F^\varepsilon))$, that is for all $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \psi(v,\tilde{v}) G^\varepsilon(t, dv, d\tilde{v}) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \psi(v,\tilde{v}) F^\varepsilon(0, dv, d\tilde{v})$$
$$+ \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G^\varepsilon(s, dv, d\tilde{v}) \int_{\mathbb{R}^3\times\mathbb{R}^3} F^\varepsilon(s, dv_*, d\tilde{v}_*) \mathcal{A}\psi(v, v_*, \tilde{v}, \tilde{v}_*). \quad (3.14)$$

For each $t \geq 0$, we denote by $g^\varepsilon(t, dv) = \int_{\tilde{v}\in\mathbb{R}^3} G^\varepsilon(t, dv, d\tilde{v})$ and $\tilde{g}^\varepsilon(t, d\tilde{v}) = \int_{v\in\mathbb{R}^3} G^\varepsilon(t, dv, d\tilde{v})$ the marginals of $G^\varepsilon(t)$. Using Lemma 3.3-(i) and that $F^\varepsilon \in \mathcal{H}(f(t), \tilde{f}(t))$, one easily checks that g_ε satisfies (3.10). Indeed, for $\phi \in Lip(\mathbb{R}^3)$, applying (3.14) to $\psi(v, \tilde{v}) = \phi(v)$ gives (3.10). Hence, by Proposition 3.4, $g^\varepsilon \equiv f$. By the same way, $\tilde{g}^\varepsilon \equiv \tilde{f}$, so that for each $t \geq 0$, $G^\varepsilon(t)$ belongs to $\mathcal{H}(f(t), \tilde{f}(t))$.

Step 2. We set $u^\varepsilon(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |v - \tilde{v}| G^\varepsilon(t, dv, d\tilde{v})$, and we apply (3.14) with the choice $\psi(v, \tilde{v}) = |v - \tilde{v}|$. Using Lemma 3.3-(ii), we obtain, setting $\kappa_0 = 2\pi\kappa_1(2\kappa_2 + \kappa_3)$ and using (3.13),

$$u^\varepsilon(t) \leq \int_{\mathbb{R}^3} |v - \tilde{v}| F^\varepsilon(0, dv, d\tilde{v})$$
$$+\kappa_0 \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G^\varepsilon(s, dv, d\tilde{v}) \int_{\mathbb{R}^3} F^\varepsilon(s, dv_*, d\tilde{v}_*)\{|v - \tilde{v}| + |v_* - \tilde{v}_*|\}$$
$$\leq d_1(f(0), \tilde{f}(0)) + \varepsilon + \kappa_0 \int_0^t ds \left[u^\varepsilon(s) + d_1(f(s), \tilde{f}(s)) + \varepsilon\right]. \quad (3.15)$$

Since $G^\varepsilon \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$, the Gronwall Lemma allows us to conclude that for all $t \geq 0$,

$$u^\varepsilon(t) \leq \left[d_1(f(0), \tilde{f}(0)) + \varepsilon(1 + \kappa_0 t) + \kappa_0 \int_0^t d_1(f(s), \tilde{f}(s)) ds\right] e^{\kappa_0 t}. \quad (3.16)$$

Step 3. Due to Step 1, we know that for each $t \geq 0$, $\varepsilon > 0$, $G^\varepsilon(t) \in \mathcal{H}(f(t), \tilde{f}(t))$, so that $d_1(f(t), \tilde{f}(t)) \leq u^\varepsilon(t)$. We thus obtain, making ε tend to 0 in (3.16) and taking the supremum over time, that for all $t \geq 0$,

$$\sup_{[0,t]} d_1(f(s), \tilde{f}(s)) \leq \left[d_1(f(0), \tilde{f}(0)) + \kappa_0 t \sup_{[0,t]} d_1(f(s), \tilde{f}(s))\right] e^{\kappa_0 t}. \quad (3.17)$$

We set $u(t) = \sup_{[0,t]} d_1(f(s), \tilde{f}(s))$. Noting that for $x \in [0, 1/4]$, $xe^x \leq 1/2$ and $1/(1 - xe^x) \leq e^{2x}$, we obtain, for $t \in [0, 1/4\kappa_0]$,

$$u(t) \leq d_1(f(0), \tilde{f}(0)) \frac{e^{\kappa_0 t}}{1 - \kappa_0 t e^{\kappa_0 t}} \leq d_1(f(0), \tilde{f}(0)) e^{3\kappa_0 t}. \quad (3.18)$$

We thus have proved that for any pair of weak solutions $f, \tilde{f} \in L^\infty_{loc}(\mathcal{P}^d_1(\mathbb{R}^3))$ to (1.1), for all $t \in [0, 1/4\kappa_0]$,

$$d_1(f(t), \tilde{f}(t)) \leq d_1(f(0), \tilde{f}(0)) e^{3\kappa_0 t}. \quad (3.19)$$

It is not hard to iterate this estimate, since κ_0 does not depend on the initial data. For any pair of weak solutions $f, \tilde{f} \in L^\infty_{loc}(\mathcal{P}_1^d(\mathbb{R}^3))$ to (1.1), for all $t \geq 0$, find $n \in \mathbb{N}$ such that $t/n \in [0, 1/4\kappa_0]$, and notice that

$$\begin{aligned} d_1(f(t), \tilde{f}(t)) &\leq d_1(f(t-t/n), \tilde{f}(t-t/n))e^{3\kappa_0 \frac{t}{n}} \\ &\leq d_1(f(t-2t/n), \tilde{f}(t-2t/n))e^{3\kappa_0 \frac{2t}{n}} \\ &\leq \ldots \leq d_1(f(0), \tilde{f}(0))e^{3\kappa_0 t}. \end{aligned} \tag{3.20}$$

This concludes the proof, since $3\kappa_0 = 6\pi\kappa_1(2\kappa_2 + \kappa_3)$. □

4. PROOF OF TECHNICAL RESULTS

We still have to prove Lemma 3.6, Propositions 3.4 and 3.5.

Proof of Lemma 3.6. We consider two probability densities g and $\tilde{g}$, and $H \in \mathcal{H}(g, \tilde{g})$. We fix $\varepsilon \in (0, 1)$. We have to show that there exists $G \in \mathcal{H}_d(g, \tilde{g})$ such that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|G(v,\tilde{v})dvd\tilde{v} \leq \int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|H(dv,d\tilde{v}) + \varepsilon. \tag{4.1}$$

We disintegrate $H(dv, d\tilde{v}) = g(v)dv\ \alpha(v, d\tilde{v})$. We also consider a partition $\mathbb{R}^3 = \cup_{i\in\mathbb{N}} A_i$, enjoying the following property: for each $i \in \mathbb{N}$, each $v, \tilde{v} \in A_i$, $|v - \tilde{v}| \leq \varepsilon$. We set $\alpha_i = \int_{A_i} \tilde{g}(\tilde{v})d\tilde{v}$ and

$$G(v,\tilde{v}) := g(v)\int_{w\in\mathbb{R}^3} \alpha(v,dw) \sum_{i\in\mathbb{N}} \frac{1}{\alpha_i} 1\!\!1_{\{w\in A_i\}} 1\!\!1_{\{\tilde{v}\in A_i\}} g(\tilde{v}). \tag{4.2}$$

Easy computations, using that for all v, $\int_{\mathbb{R}^3} \alpha(v, d\tilde{v}) = 1$ while for all $\tilde{v}$, $\int_{v\in\mathbb{R}^3} g(v)dv\ \alpha(v, d\tilde{v}) = g(\tilde{v})d\tilde{v}$, show that $G(v, \tilde{v})dvd\tilde{v}$ belongs to $\mathcal{H}_d(g, \tilde{g})$. Finally,

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde{v}|G(v,\tilde{v})dvd\tilde{v} \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-\tilde{v}|g(v)\alpha(v,dw)\sum_{i\in\mathbb{N}}\frac{1}{\alpha_i} 1\!\!1_{\{w\in A_i\}} 1\!\!1_{\{\tilde{v}\in A_i\}} g(\tilde{v})dvd\tilde{v} \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (|v-w| + |w-\tilde{v}|)\, g(v)\alpha(v,dw) \\ &\quad \times \sum_{i\in\mathbb{N}}\frac{1}{\alpha_i} 1\!\!1_{\{w\in A_i\}} 1\!\!1_{\{\tilde{v}\in A_i\}} g(\tilde{v})dvd\tilde{v} \end{aligned}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v - w| g(v) dv \, \alpha(v, dw)$$
$$+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |w - \tilde{v}| \tilde{g}(w) dw \sum_{i\in\mathbb{N}} \frac{1}{\alpha_i} 1\!\!1_{\{w\in A_i\}} 1\!\!1_{\{\tilde{v}\in A_i\}} g(\tilde{v}) d\tilde{v}$$
$$\leq \int_{\mathbb{R}^3\times\mathbb{R}^3} |v - w| H(dv, dw)$$
$$+ \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{g}(w) dw \sum_{i\in\mathbb{N}} \frac{1}{\alpha_i} 1\!\!1_{\{w\in A_i\}} 1\!\!1_{\{\tilde{v}\in A_i\}} g(\tilde{v}) d\tilde{v} \tag{4.3}$$

since for each i, each $w, \tilde{v} \in A_i$, $|w - \tilde{v}| \leq \varepsilon$. This last term equals $\int |v - w| H(dv, dw) + \varepsilon$, so that (4.1) holds. □

Proof of Proposition 3.5. We fix $F \in L^\infty_{loc}(\mathcal{P}_1^d(\mathbb{R}^3 \times \mathbb{R}^3))$, and we denote, for each $t \geq 0$, each $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\mathcal{E}_t\psi(v, \tilde{v}, \theta, \varphi)$$
$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} F(t, dv_*, d\tilde{v}_*) \left\{ \begin{array}{l} [b(|v - v_*|) \wedge b(|\tilde{v} - \tilde{v}_*|)] [\psi(v', \tilde{v}') - \psi(v, \tilde{v})] \\ + [b(|v - v_*|) - b(|\tilde{v} - \tilde{v}_*|)]_+ [\psi(v', \tilde{v}) - \psi(v, \tilde{v})] \\ + [b(|\tilde{v} - \tilde{v}_*|) - b(|v - v_*|)]_+ [\psi(v, \tilde{v}') - \psi(v, \tilde{v})] \end{array} \right\}. \tag{4.4}$$

We have to show the existence of $G \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$ such that for all $t \geq 0$, $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} G(t, dv, d\tilde{v}) \psi(v, \tilde{v}) = \int_{\mathbb{R}^3\times\mathbb{R}^3} F(0, dv, d\tilde{v}) \psi(v, \tilde{v})$$
$$+ \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G(s, dv, d\tilde{v}) \int_0^\pi \beta(\theta) d\theta \int_0^{2\pi} d\varphi \mathcal{E}_s \psi(v, \tilde{v}, \theta, \varphi). \tag{4.5}$$

We endow $\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ with the weak convergence topology, which can be metrized by

$$\delta(H, \tilde{H}) = \sup_{\phi\in BL_1} \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi [dH - d\tilde{H}] \right|, \tag{4.6}$$

BL_1 standing for the set of globally Lipschitz functions from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}$, bounded by 1 and with a Lipschitz constant smaller than 1.

Step 1. Assume first that in addition to (H), $\Lambda = \int_0^\pi \beta(\theta) d\theta < \infty$. Denote by $\mathcal{C}(\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)) := \mathcal{C}([0, \infty), \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3))$. Then one easily checks the existence (and uniqueness) of $G = (G(t))_{t\geq 0} \in \mathcal{C}(\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3))$, satisfying (4.5) for all bounded measurable function ψ. Indeed, it follows from classical arguments, remarking that for any bounded measurable function $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$,

$$\sup_{s\in[0,\infty)} \left|\left| \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \mathcal{E}_s \psi(.,\theta,\varphi) \right|\right|_\infty \leq 12\pi\Lambda\kappa_2 ||\psi||_\infty. \tag{4.7}$$

Using that $F \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$ and (H), one easily checks that G belongs to $L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$, and that (4.5) actually holds for any measurable function bounded by $C(1+|v|)$.

Step 2. We thus consider, for each $\varepsilon \in (0,1)$, $\beta_\varepsilon(\theta) = \beta(\theta)\mathbb{1}_{\{\theta\geq\varepsilon\}}$. Then $\Lambda_\varepsilon = \int_0^\pi \beta_\varepsilon(\theta)d\theta < \infty$, so that due to Step 1, there exists $(G_\varepsilon(t))_{t\geq 0} \in \mathcal{C}(\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$, satisfying (4.5) with β_ε instead of β. We now show that for all $T \geq 0$,

$$K_T := \sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} m_1(G^\varepsilon(t)) < \infty \text{ where}$$

$$m_1(G^\varepsilon(t)) := \int_{\mathbb{R}^3\times\mathbb{R}^3} (|v| + |\tilde{v}|) G^\varepsilon(t, dv, d\tilde{v}). \tag{4.8}$$

We use (4.5) with $\psi(v,\tilde{v}) = |v| + |\tilde{v}|$. With the help of (H), we obtain

$$\begin{aligned} m_1(G^\varepsilon(t)) &\leq m_1(F(0)) \\ &\quad + \kappa_2 \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G^\varepsilon(s, dv, d\tilde{v}) \int_{\mathbb{R}^3\times\mathbb{R}^3} F(s, dv_*, d\tilde{v}_*)\Delta_\varepsilon, \end{aligned} \tag{4.9}$$

where, thanks to (2.6),

$$\begin{aligned} \Delta_\varepsilon &= \int_\varepsilon^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi ||v'| - |v| + |\tilde{v}'| - |\tilde{v}|| \\ &\leq \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \frac{\theta}{2} (|v - v_*| + |\tilde{v} - \tilde{v}_*|) \\ &\leq \pi\kappa_1 (|v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|). \end{aligned} \tag{4.10}$$

We finally obtain

$$m_1(G^\varepsilon(t)) \leq m_1(F(0)) + \pi\kappa_1\kappa_2 \int_0^t ds \left[m_1(G^\varepsilon(s)) + m_1(F(s))\right]. \tag{4.11}$$

Since $F \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$ by assumption, the Gronwall Lemma allows us to conclude that (4.8) holds.

Step 3. We now prove an *equicontinuity* result, namely that for all $T \geq 0$,

$$\lim_{h\to 0} \sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} \delta(G^\varepsilon(t), G^\varepsilon(t+h)) = 0. \tag{4.12}$$

To do so, we consider $\psi \in BL_1$ and notice that for all $0 \le t \le T$, due to (H), (2.6) and since $F \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$,

$$
\begin{aligned}
|\mathcal{E}_t\psi(v,\tilde v,\theta,\varphi)| &\le 3\kappa_2 \int_{\mathbb{R}^3\times\mathbb{R}^3} F(t,dv_*,d\tilde v_*)[|v'-v| + |\tilde v' - \tilde v|] \\
&\le 3\kappa_2 \frac{\theta}{2} \int_{\mathbb{R}^3\times\mathbb{R}^3} F(t,dv_*,d\tilde v_*)\,[|v-v_*| + |\tilde v - \tilde v_*|] \\
&\le 3\kappa_2 \frac{\theta}{2}\,[|v| + |\tilde v| + m_1(F(t))] \le 3\kappa_2 \frac{\theta}{2}\,[|v| + |\tilde v| + C_T], \quad (4.13)
\end{aligned}
$$

with $C_T = \sup_{[0,T]} m_1(F(t))$. Hence for all $0 \le t \le t+h \le T$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} \psi\,[dG^\varepsilon(t+h) - dG^\varepsilon(t)]\right| \\
&= \left|\int_t^{t+h} ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G^\varepsilon(s,dv,d\tilde v) \int_0^\pi \beta_\varepsilon(\theta)d\theta \int_0^{2\pi} d\varphi\mathcal{E}_s\psi(v,\tilde v,\theta,\varphi)\right| \\
&\le 3\pi\kappa_1\kappa_2 \int_t^{t+h} ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G^\varepsilon(s,dv,d\tilde v)\,[|v| + |\tilde v| + C_T] \\
&\le 3\pi\kappa_1\kappa_2\,[C_T + K_T]\,h, \qquad (4.14)
\end{aligned}
$$

where K_T was defined in (4.8). One immediately concludes that (4.12) holds.

Step 4. By the Arleza-Ascoli Theorem (using (4.8) and (4.12)), we may find a sequence $\varepsilon_n \to 0$ such that $G_n := G^{\varepsilon_n}$ goes to some $G = (G(t))_{t\ge 0}$ in $\mathcal{C}(\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3))$ as n tends to infinity. We deduce from (4.8) that:

(i) G belongs to $L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3 \times \mathbb{R}^3))$,
(ii) for any $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ continuous and bounded by $C(1 + |v| + |\tilde v|)$, any $t \ge 0$, $\lim_n \int \psi dG_n(t) = \int \psi dG(t)$,
(iii) for any $T \ge 0$, any measurable function $\Gamma : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ with $(v,\tilde v) \mapsto \Gamma(s,v,\tilde v)$ continuous for each s and such that $|\Gamma(s,v,\tilde v)| \le C(1 + |v| + |\tilde v|)$,

$$
\lim_n \int_0^T ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G_n(s,dv,d\tilde v)\Gamma(s,v,\tilde v) = \int_0^T ds \int_{\mathbb{R}^3\times\mathbb{R}^3} G(s,dv,d\tilde v)\Gamma(s,v,\tilde v). \qquad (4.15)
$$

It remains to take limits in (4.5). Let thus $t \ge 0$ and $\psi \in Lip(\mathbb{R}^3 \times \mathbb{R}^3)$. Due to (ii), $\int_{\mathbb{R}^3\times\mathbb{R}^3} \psi dG_n(t)$ tends to $\int_{\mathbb{R}^3\times\mathbb{R}^3} \psi dG(t)$ ad n tends to infinity, so that we just have to prove that

$$\int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} (dG_n(s) - dG(s)) \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \mathcal{E}_s\psi(.,\theta,\varphi)$$
$$- \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} dG_n(s) \int_0^\pi 1\!\!1_{\{\theta\le\varepsilon_n\}}\beta(\theta)d\theta \int_0^{2\pi} d\varphi \mathcal{E}_s\psi(.,\theta,\varphi) =: A_n - B_n \tag{4.16}$$

tends to 0 as n tends to infinity. First, using that $|\mathcal{E}_s\psi(v,\tilde{v},\theta,\varphi)| \le \theta C(T,F,\psi)(1+|v|+|\tilde{v}|)$ for all $0\le s\le t\le T$ (see (4.13)) and (4.8), we obtain

$$|B_n| \le 2\pi T C(T,F,\psi)(1+K_T)\int_0^{\varepsilon_n} \theta\beta(\theta)d\theta \tag{4.17}$$

which tends to 0 due to (H). Next, point (iii) above shows that $\lim_n A_n = 0$. Indeed, we already have seen that $|\int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi\mathcal{E}_s\psi(v,\tilde{v},\theta,\varphi)| \le 2\pi\kappa_1 C(T,F,\psi)(1+|v|+|\tilde{v}|)$. On the other hand, one may check that $(v,\tilde{v})\mapsto \int_0^\pi \beta(\theta)d\theta\int_0^{2\pi} d\varphi\mathcal{E}_s\psi(v,\tilde{v},\theta,\varphi)$ is continuous for each $s\in[0,t]$: it follows from the Lebesgue dominated convergence Theorem, the facts that $F(s)$ has a density and that for all $\theta\in(0,\pi]$, all $\varphi\in[0,2\pi]$, almost all $v_*, \tilde{v}_*$, $(v,\tilde{v})\mapsto(v',\tilde{v}')$ is continuous (recall Notation 3.1 and Lemma 3.2).

Thus G satisfies (4.5) for any globally Lipschitz function. □

Remark that we use only in the very end of this auxilliary proof the assumption that f, $\tilde{f}$ are solutions admitting densities.

Proof of Proposition 3.4. Let $f, g \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ be as in the statement. A formal computation shows that $\frac{d}{dt}\int_{\mathbb{R}^3}|f(t)-g(t)|(dv)\le 0$, from which the result would follow. However, we would have to apply (3.10) with the test function $\phi(t,v) = sign(f(t,v)-g(t,v))$, which is not Lipschitz, so that the computation is not licit. We have not been able to make this calculus rigorous.

We will use some martingale problems techniques. We consider a weak solution $f=(f(t))_{t\ge0}\in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ to (1.1). We also consider, for each $t\ge0$ the operator A_t defined for $\phi\in Lip(\mathbb{R}^3)$ and $v\in\mathbb{R}^3$ by

$$A_t\phi(v) = \int_{\mathbb{R}^3} f(t,dv_*)A\phi(v,v_*). \tag{4.18}$$

We will prove that for any $\mu\in\mathcal{P}_1(\mathbb{R}^3)$, there exists at most one $g\in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ such that for all $t\ge0$, all $\phi\in Lip_b(\mathbb{R}^3)$ (the set of globally Lipschitz bounded functions)

$$\int_{\mathbb{R}^3}\phi(v)g(t,dv) = \int_{\mathbb{R}^3}\phi(v)\mu(dv) + \int_0^t ds\int_{\mathbb{R}^3} g(s,dv)A_s\phi(v). \tag{4.19}$$

Since by assumption, f and g solve this equation with $\mu = f(0)$, this will conclude the proof.

Step 1. Let $\mu \in \mathcal{P}_1(\mathbb{R}^3)$. A càdlàg adapted $\mathbb{R}^3$-valued stochastic process $(V_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is said to solve the martingale problem $MP((A_t)_{t\geq 0}, \mu, Lip_b(\mathbb{R}^3))$ if $P \circ V_0^{-1} = \mu$ and if for all $\phi \in Lip_b(\mathbb{R}^3)$, $(M_t^\phi)_{t\geq 0}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$-martingale, where

$$M_t^\phi = \phi(V_t) - \int_0^t A_s\phi(V_s)ds. \tag{4.20}$$

Assume for a moment that:

(i) there exists a countable subset $(\phi_k)_{k\geq 1} \subset Lip_b(\mathbb{R}^3)$ such that for all $t \geq 0$, the closure of $\{(\phi_k, A_t\phi_k), k \geq 1\}$ (for the bounded pointwise convergence) contains $\{(\phi, A_t\phi), \phi \in Lip_b(\mathbb{R}^3)\}$,
(ii) for each $v_0 \in \mathbb{R}^3$, there exists a solution to $MP((A_t)_{t\geq 0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$,
(iii) for each $v_0 \in \mathbb{R}^3$, uniqueness (in law) holds for $MP((A_t)_{t\geq 0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$,

Then, due to Bhatt-Karandikar [2, Theorem 5.2] (see also Remark 3.1 and Theorem 5.1 in ref. 2 and Theorem B.1 in Ref. 8), uniqueness for (4.19) holds. First, it is immediate that (i) holds, considering a countable subset $(\phi_k)_{k\geq 1} \subset Lip_b(\mathbb{R}^3)$ dense in $Lip_b(\mathbb{R}^3)$ for the norm $|||\phi||| = ||\phi||_\infty + \sup_{x\neq y}\left|\frac{\phi(x)-\phi(y)}{x-y}\right|$, using (2.6) and the fact that $f \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$.

Step 2. Classical arguments (see e.g. Tanaka [11, Section 4] or Desvillettes-Graham-Méléard [4, Theorem 3.8]) yield that a process $(V_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a solution to $MP((A_t)_{t\geq 0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$ if and only if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t\geq 0}$-adapted Poisson measure $N(dt, dv_*, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi) \times (0, 2\pi) \times [0, \kappa_2]$ with intensity measure $dtf(t, dv_*)\beta(\theta)d\theta d\varphi du$ such that (recall that a was defined in Notation 3.1)

$$V_t = v_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^\pi \int_0^{2\pi} \int_0^{\kappa_2} a(V_{s-}, v_*, \theta, \varphi) 1\!\!1_{\{u\leq b(|V_{s-}-v_*|)\}} N(ds, dv_*, d\theta, d\varphi, du). \tag{4.21}$$

We thus just have to prove the existence and uniqueness in law for solutions to (4.21). After some preliminary stated in Step 3, we will prove the uniqueness result in Steps 4,5,6 and study the existence in Step 7.

Step 3. We will use that for $N = \sum_{i\geq 1} \delta_{(T_i, v_i, \theta_i, \varphi_i, u_i)}$ a Poisson measure as in Step 2, and for $\hat{\varphi} : \Omega \times \mathbb{R}^3 \times (0, \pi) \times (0, 2\pi) \mapsto (0, 2\pi)$ a predictable function, $\hat{N} = \sum_{i\geq 1} \delta_{(T_i, v_i, \theta_i, \varphi_i + \hat{\varphi}(T_i, v_i, \theta_i, \varphi_i), u_i)}$ is also a Poisson measure as in Step 2. Such a property was noted by Tanaka,[10] see also [6, Lemma 4.7].

Step 4. For $k \geq 1$ and N as in Step 2, the Poisson measure $1\!\!1_{\{\theta>1/k\}} 1\!\!1_{\{s\leq T\}} N(ds, dv_*, d\theta, d\varphi, du)$ is a.s. finite for all $T > 0$, so that there obviously exists a unique (necessarily càdlàg and adapted) solution to

$$V_t^k = v_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^\pi \int_0^{2\pi} \int_0^{\kappa_2} a(V_{s-}^k, v_*, \theta, \varphi) \\ \times 1\!\!1_{\{u \leq b(|V_{s-}^k - v_*|)\}} 1\!\!1_{\{\theta>1/k\}} N(ds, dv_*, d\theta, d\varphi, du). \tag{4.22}$$

Furthermore, the law of $(V_t^k)_{t\geq 0}$ does not depend on the probability space nor on the Poisson measure N.

Step 5. We check that for $(V_t)_{t\geq 0}$ a solution to (4.21), $(V_t^k)_{k\geq 0}$ a solution to (4.22), for all $T > 0$, using (2.6),

$$C_T := E\left[\sup_{[0,T]} |V_t|\right] + \sup_{k\geq 1} E\left[\sup_{[0,T]} |V_t^k|\right] < \infty. \tag{4.23}$$

We thus consider, for each $n \geq 1$, the stopping time $\tau_n = \inf\{t \geq 0,\ |V_t| \geq n\}$. Using (2.6), we deduce that for all $0 \leq t \leq T$,

$$\begin{aligned} E&\left[\sup_{[0,t\wedge\tau_n]} |V_s|\right] \\ &\leq |v_0| + E\left[\int_0^{t\wedge\tau_n} \int_{\mathbb{R}^3} f(s, dv_*) \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi b(|V_{s-} - v_*|)|a(V_{s-}, v_*, \theta, \varphi)|\right] \\ &\leq |v_0| + E\left[\int_0^{t\wedge\tau_n} \int_{\mathbb{R}^3} f(s, dv_*) \int_0^\pi \beta(\theta)d\theta \int_0^{2\pi} d\varphi \kappa_2 \frac{\theta}{2} |V_{s-} - v_*|\right] \\ &\leq |v_0| + \pi\kappa_1\kappa_2 T \sup_{[0,T]} m_1(f(s)) + \pi\kappa_1\kappa_2 \int_0^t E\left[\sup_{[0,s\wedge\tau_n]} |V_t|\right]. \end{aligned} \tag{4.24}$$

Let thus $C_T := [|v_0| + \pi\kappa_1\kappa_2 T \sup_{[0,T]} m_1(f(s))]e^{\pi\kappa_1\kappa_2 T}$. The Gronwall Lemma allows us to conclude that for all $T \geq 0$, $\sup_n E\left[\sup_{[0,T\wedge\tau_n]} |V_t|\right] \leq C_T$, from which it readily follows that a.s. $\lim_n \tau_n = \infty$, so that finally, $E\left[\sup_{[0,T]} |V_t|\right] \leq C_T$. The same computation works for $(V_t^k)_{t\geq 0}$, so that (4.23) holds.

Step 6. Let $(V_t)_{t\geq 0}$ be a càdlàg adapted solution to (4.21) with some Poisson measure N as in Step 2. Recall Lemma 3.2, and define $(\tilde{V}_t^k)_{t\geq 0}$ as the solution (which clearly exists and is unique since $1\!\!1_{\{\theta>1/k\}} 1\!\!1_{\{s\leq T\}} N(ds, dv_*, d\theta, d\varphi, du)$ is a.s. finite for all $T > 0$) to

$$\tilde{V}_t^k = v_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\kappa 2} a\big(\tilde{V}_{s-}^k, v_*, \theta, \varphi + \varphi_0\big(V_{s-} - v_*, \tilde{V}_{s-}^k - v_*\big)\big) \times \mathbb{1}_{\{u \leq b(|\tilde{V}_{s-}^k - v_*|)\}} \mathbb{1}_{\{\theta > 1/\kappa\}} N(ds, dv_*, d\theta, d\varphi, du). \tag{4.25}$$

The map $\varphi_0(V_{s-} - v_*, \tilde{V}_{s-}^k - v_*)$ being predictable, we deduce from Steps 3 and 4 that $(\tilde{V}_t^k)_{t \geq 0}$ has the same law as $(V_t^k)_{t \geq 0}$. We will now show that $(\tilde{V}_t^k)_{t \geq 0}$ goes in probability to $(V_t)_{t \geq 0}$, which will yield the uniqueness of the law of $(V_t)_{t \geq 0}$ and thus will end the proof of (iii). First,

$$\sup_{[0,t]} |V_s - \tilde{V}_s^k| \leq \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\kappa 2} \Delta_k(s, v_*, \theta, \varphi, u) N(ds, dv_*, d\theta, d\varphi, du), \tag{4.26}$$

with

$$\Delta_k = \big|a(V_{s-}, v_*, \theta, \varphi) \mathbb{1}_{\{u \leq b(|V_{s-} - v_*|)\}} - a\big(\tilde{V}_{s-}^k, v_*, \theta, \varphi + \varphi_0\big(V_{s-} - v_*, \tilde{V}_{s-}^k - v_*\big)\big) \mathbb{1}_{\{u \leq b(|\tilde{V}_{s-}^k - v_*|)\}} \mathbb{1}_{\{\theta > 1/k\}}\big|. \tag{4.27}$$

Taking the expectation, we get

$$E\left[\sup_{[0,t]} |V_s - \tilde{V}_s^k|\right] \leq \int_0^t ds \int_{\mathbb{R}^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\kappa_2} f(s, dv_*) \beta(\theta) d\theta d\varphi du E\left[\Delta_k(s, v_*, \theta, \varphi, u)\right]. \tag{4.28}$$

Integrating in u over $[0, \kappa_2]$, we may upperbound the right hand side of (4.28) by

$$\int_0^t ds \int_{\mathbb{R}^3} f(s, dv_*) \int_0^{\pi} \beta(\theta) d\theta \int_0^{2\pi} d\varphi E\Big[\big\{ b\big(\big|\tilde{V}_{s-}^k - v_*\big|\big) \wedge b(|V_{s-} - v_*|) \big\} \big|a\big(V_{s-}, v_*, \theta, \varphi\big) - a\big(\tilde{V}_{s-}^k, v_*, \theta, \varphi + \varphi_0\big(V_{s-} - v_*, \tilde{V}_{s-}^k - v_*\big)\big)\big| \Big]$$
$$+ \int_0^t ds \int_{\mathbb{R}^3} f(s, dv_*) \int_0^{\pi} \beta(\theta) d\theta \int_0^{2\pi} d\varphi E\Big[\big\{ b\big(\big|\tilde{V}_s^k - v_*\big|\big) - b(|V_s - v_*|) \big\}_+ \times \big|a\big(\tilde{V}_{s-}^k, v_*, \theta, \varphi + \varphi_0\big(V_{s-} - v_*, \tilde{V}_{s-}^k - v_*\big)\big)\big| \Big]$$
$$+ \int_0^t ds \int_{\mathbb{R}^3} f(s, dv_*) \int_0^{\pi} \beta(\theta) d\theta \int_0^{2\pi} d\varphi E\Big[\big\{ b(|V_{s-} v_*|) - b\big(\big|\tilde{V}_{s-}^k - v_*\big|\big) \big\}_+ \times |a(V_{s-}, v_*, \theta, \varphi)| \Big]$$

$$+\int_0^t ds \int_{\mathbb{R}^3} f(s, dv_*) \int_0^{1/k} \beta(\theta) d\theta \int_0^{2\pi} d\varphi E\Big[b\big(\big|\tilde{V}^k_{s-} - v_*\big|\big)\Big)$$
$$\times \big|a\big(\tilde{V}^k_{s-}, v_*, \theta, \varphi + \varphi_0\big(V_{s-} - v_*, \tilde{V}^k_{s-} - v_*\big)\big)\big| \Big] \quad (4.29)$$

Using now (H), (2.6) and Lemma 3.2, a calculus (as in the proof of Lemma 3.3-(ii)) gives

$$E\left[\sup_{[0,t]} \left|V_s - \tilde{V}^k_s\right|\right] \le 2\pi\kappa_1(2\kappa_2 + \kappa_3) \int_0^t ds E\left[\left|V_s - \tilde{V}^k_s\right|\right]$$
$$+ \pi\kappa_2 \int_0^{1/k} \theta\beta(\theta) d\theta \int_0^t ds \left[E\left|\tilde{V}^k_s\right| + \int_{\mathbb{R}^3} |v_*| f(s, dv_*)\right]. \quad (4.30)$$

Using finally that $f \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$ and (4.23), using the Gronwall Lemma (licit due to (4.23)) and the fact that $\lim_k \int_0^{1/k} \theta\beta(\theta)d\theta = 0$ (due to (H)) we get, for all $T \ge 0$,

$$\lim_k E\left[\sup_{[0,T]} \left|V_s - \tilde{V}^k_s\right|\right] = 0, \quad (4.31)$$

which concludes the uniqueness proof for $MP((A_t)_{t\ge0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$.

Step 7. It remains to prove the existence for $MP((A_t)_{t\ge0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$. We use to this aim a Picard iteration. Let N be a Poisson measure as in Step 2. We consider the constant process $V^0_t \equiv v_0$, we set $\varphi^*_0 = 0$ and we define inductively

$$V^{n+1}_t = v_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^\pi \int_0^{2\pi} \int_0^{\kappa_2} a\big(V^n_{s-}, v_*, \theta, \varphi + \varphi^*_n(s, v_*)\big) 1\!\!1_{\{u \le b(|V^n_{s-} v_*|)\}}$$
$$\times N(ds, dv_*, d\theta, d\varphi, du), \quad (4.32)$$

and $\varphi^*_{n+1}(s, v_*) = \varphi^*_n(s, v_*) + \varphi_0(V^{n+1}_{s-} - v_*, V^n_{s-} - v_*)$ (recall Lemma 3.2). One easily checks that (4.32) is well-defined due to (H) and (2.6). A computation as in Step 6 yields that for all $t \ge 0$, all $n \ge 0$,

$$E\left[\sup_{[0,t]} \left|V^{n+1}_s - V^n_s\right|\right] \le 2\pi\kappa_1(2\kappa_2 + \kappa_3) \int_0^t ds E\big[\big|V^n_s - V^{n-1}_s\big|\big], \quad (4.33)$$

so that there exists a càdlàg adapted process $(V_t)_{t\ge0}$ such that

$$\lim_n E\left[\sup_{[0,T]} \left|V_s - V^n_s\right|\right] = 0 \text{ and } E\left[\sup_{[0,T]} \left|V_s\right|\right] < \infty \quad (4.34)$$

for all $T \geq 0$. To show that $(V_t)_{t\geq 0}$ satisfies $MP((A_t)_{t\geq 0}, \delta_{v_0}, Lip_b(\mathbb{R}^3))$, we need to check that for all $0 \leq s_1 \leq ... \leq s_l \leq s \leq t \leq T$, all $\phi_1, ..., \phi_l \in C_b(\mathbb{R}^3)$, and all $\phi \in Lip_b(\mathbb{R}^3)$,

$$E\left[\left(\prod_{i=1}^{l} \phi_i\left(V_{s_i}\right)\right)\left(\phi(V_t) - \phi(V_s) - \int_s^t A_u\phi(V_u)du\right)\right] = 0. \qquad (4.35)$$

But we know from (4.32) that for all $n \geq 1$,

$$E\left[\left(\prod_{i=1}^{l} \phi_i\left(V_{s_i}^n\right)\right)\left(\phi\left(V_t^{n+1}\right) - \phi\left(V_s^{n+1}\right) - \int_s^t A_u\phi\left(V_u^n\right)du\right)\right] = 0. \qquad (4.36)$$

Since $v \mapsto A_u\phi(v)$ is continuous (by Lemma 3.2 for example) and bounded by $C_T(1+|v|)$ (due to (2.6), (H) and since $f \in L^\infty_{loc}(\mathcal{P}_1(\mathbb{R}^3))$), we obtain (4.35) by going to the limit in (4.36) using (4.34). □

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